

Review

Structure sheaf

$\mathcal{X}$ DM stack.

- $\mathcal{O}_{\mathcal{X}}(U) = T(U, \mathcal{O}_U) \quad (\forall U \text{ étale } \mathcal{X}\text{-scheme})$

- extending sections: $\mathcal{O}_{\mathcal{X}}(\mathcal{U}) = E_{\mathcal{U}}(\mathcal{O}_{\mathcal{X}}(U) \Rightarrow \mathcal{O}_{\mathcal{X}}(R)) \quad (\forall U \text{ étale } \mathcal{X}\text{-DM stack})$
 $(U \rightarrow \mathcal{X} \text{ étale presentation, } R = U \times_{\mathcal{X}} U)$

G finite group $\curvearrowright \text{Spec } A$ G -equivariant algebra

• $A^G \rightarrow A$ integral. If A f.g. / Noetherian ring R , $A^G \rightarrow A$ finite & A^G f.g. / R .

• $A^G \rightarrow B$ ring homo. $G \curvearrowright \text{Spec}(B \otimes_{A^G} A)$. $\text{Spec}(B \otimes_{A^G} A)^G \rightarrow \text{Spec } B$
If $A^G \rightarrow B$ flat, then $B = (B \otimes_{A^G} A)^G$ universal homo

Local Structure Theorem

 Theorem. $\mathcal{X}$ separated DM stack, $x \in |\mathcal{X}|$ w/ geometric stabilizer G_x
 $\exists$ affine étale morphism $f: ([\text{Spec } A/G_x], w) \rightarrow (\mathcal{X}, x)$
s.t. induces iso of geom stabilizer groups at w .

Today: Keel - Mori Theorem

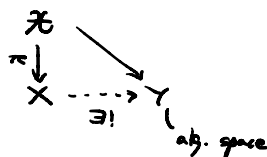
(existence of coarse moduli spaces)

Def. $\pi: \mathcal{X} \rightarrow X$ is a coarse moduli space (c.m.s.) if

$\downarrow$ $\downarrow$
 alg. stack alg. space

(1) $\forall k = \overline{k}: \mathcal{X}(k) / \sim \xrightarrow{(\text{bij.})} X(k)$

(2) π is universal.



(verbally: motivation)

Obs. (1) G finite group $\curvearrowright$ alg. space U .

$[U/G] \rightarrow X$ c.m.s. $\Leftrightarrow U \rightarrow X$ geometric quotient

(2) $\pi: \mathcal{X} \rightarrow X$, $\{X_i \rightarrow X\}$ étale covering of X .

$\downarrow$ $\downarrow$
 alg. stack alg. space

$\forall i: \mathcal{X} \times_X X_i \rightarrow X_i$ c.m.s. $\Rightarrow \pi: \mathcal{X} \rightarrow X$ c.m.s.

Desirable properties of c.m.s.

(a) stable under fibre base change

(b) $\pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$

(c) π is proper

(d) π is universal homeo

Lemma. (a) $\Rightarrow$ (b)

pf. $\forall$ étale map $X' \rightarrow X$:

$\mathcal{X}' = \mathcal{X} \times_X X' \rightarrow X'$ c.m.s. ($\because$ (a))

$\Rightarrow \text{Mor}(X', \Lambda') \rightarrow \text{Mor}(\mathcal{X}', \Lambda')$ bij.

$\Rightarrow T(X', \mathcal{O}_{X'}) \rightarrow T(\mathcal{X}', \mathcal{O}_{\mathcal{X}})$ iso.

$\therefore \mathcal{O}_X \rightarrow \pi_* \mathcal{O}_{\mathcal{X}}$ iso.

□

Theorem. G finite group $\curvearrowright \text{Spec } A$. Consider $\pi: \text{Spec } A/G \rightarrow \text{Spec } A$.

(1) $\pi: [\text{Spec } A/G] \rightarrow \text{Spec } A^G$ is a c.m.s.

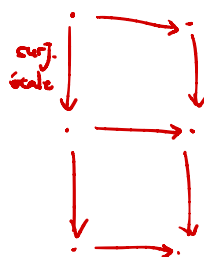
(2) Base change of π by affine $X' \rightarrow \text{Spec } A^G$ is a c.m.s.
 alg. space

(3) $X \rightarrow \pi_* \mathcal{O}_X$ is iso.

(4) If A is f.g. / Noetherian ring R , A^G is f.g. / R
 $\& \pi$ is a proper universal homeo.

Claim.

$$\begin{array}{ccc}
 \text{Spec } (B \otimes_A A) & \longrightarrow & \text{Spec } A \\
 \downarrow & & \downarrow \\
 [\text{Spec } (B \otimes_A A) / G] & \longrightarrow & [\text{Spec } A / G] \\
 \downarrow & & \downarrow \pi \\
 \text{Spec } (B \otimes_{A^G} A)^G & \square & \text{Spec } A^G \\
 \downarrow & & \\
 \text{Spec } B & \longrightarrow & \text{Spec } A^G
 \end{array}$$



top, big square $\square$
 $\Rightarrow$ bottom square $\square$

In particular, any base change of π by affine map is in the form $[\text{Spec } C / G] \rightarrow \text{Spec } C^G$.

pf. (of Theorem)

(1) $\text{Spec } A \rightarrow \text{Spec } A^G$ geometric quotient, (v)

(2) True when A' is affine & c.m.s. is a G -local property, (v)

(3) (2) & previous lemma (v)

(4) A^G f.g. / R . $\text{Spec } A \xrightarrow{\text{finite} \rightarrow \text{proper}} \text{Spec } A^G$
 $\downarrow \text{surj.} \quad \downarrow \text{proper}$
 $[\text{Spec } A / G] \rightarrow \text{Spec } A^G$

Base change by $\text{Spec } B \rightarrow \text{Spec } A^G$:

$$[\text{Spec } (B \otimes_A A) / G] \xrightarrow{\text{homeo}} \text{Spec } (B \otimes_{A^G} A)^G \xrightarrow{\text{homeo}} \text{Spec } B \quad (v) \quad \square$$

Strategy to prove Keel-Mori

$$\begin{array}{ccc}
 W \times_{\mathbb{A}^1} W \xrightarrow{\cong} W = [\mathrm{Spec} B/G] & \xrightarrow{\text{aff. ét.}} & \mathcal{X} \\
 \downarrow \text{c.m.s.} & & \downarrow \text{c.m.s.} \\
 \mathrm{Spec} B^G & \xrightarrow{\quad} & \mathrm{Spec} A^G \cdots \rightarrow ? \\
 & \uparrow & \nwarrow \text{alg. space}
 \end{array}$$

induced by G -equiv
ring homs $A \rightarrow B$

want this to be an étale equiv relation

Prop. G finite group, $f: \mathrm{Spec} A \rightarrow \mathrm{Spec} B$ G -equiv map.

A, B f.g. / Noetherian ring R .

Suppose $\pi \in \mathrm{Spec} A$ satisfies

(a) f étale at π

(b) induced map of stabilizer groups $G_\pi \rightarrow G_{f(\pi)}$ bi.

Then $\exists$ open affine nbhd $W \subset \mathrm{Spec} A^G$ of $\pi_A(\pi)$

s.t. $W \hookrightarrow \mathrm{Spec} A^G \rightarrow \mathrm{Spec} B^G$ étale and cover square of

$$\begin{array}{ccccc}
 \pi_A^{-1}(W) & \hookrightarrow & [\mathrm{Spec} A/G] & \rightarrow & [\mathrm{Spec} B/G] \\
 \downarrow & & \downarrow \pi_A & & \downarrow \pi_B \\
 W & \hookrightarrow & \mathrm{Spec} A^G & \rightarrow & \mathrm{Spec} B^G
 \end{array}$$

is cartesian.

★ Cor. Same setting & assumptions as Prop., except
 $\forall \pi \in \mathrm{Spec} A$ satisfy (a), (b).

Then $\mathrm{Spec} A^G \rightarrow \mathrm{Spec} B^G$ étale &

$$\begin{array}{ccc}
 [\mathrm{Spec} A/G] & \rightarrow & [\mathrm{Spec} B/G] \\
 \downarrow \pi_A & \square & \downarrow \pi_B \\
 \mathrm{Spec} A^G & \rightarrow & \mathrm{Spec} B^G
 \end{array}$$

pt. (of Prop.)

Step 1 Question is étale local around $\pi_B(y) \in \text{Spec } B^G$. ($y = A(x)$)

$Y' \rightarrow \text{Spec } B^G$ affine étale nbhd of $\pi_B(y)$.

$$\begin{array}{ccc} [\text{Spec } A/G] & \rightarrow & [\text{Spec } B/G] \\ \downarrow \pi_A & & \downarrow \pi_B \\ \text{Spec } A^G & \rightarrow & \text{Spec } B^G \\ \pi_A(x) & & \end{array} \quad \begin{array}{ccc} [\text{Spec } A'/G] & \rightarrow & [\text{Spec } B'/G] \\ \downarrow \pi_{A'} & & \downarrow \pi_{B'} \\ \text{Spec } A'^G & \rightarrow & \text{Spec } B'^G \\ \pi_{A'}(x') & & \end{array}$$

base change via Y'

Take $W = \text{Im}(W' \hookrightarrow \text{Spec } A'^G \rightarrow \text{Spec } A^G)$. Done by étale descent!

Step 2 Step 1 allows to take B^G strictly henselian.

(Due to $Y^{\text{sh}} = \varprojlim_{Y_\lambda \rightarrow Y} Y_\lambda$, descent of morphisms/properties under limits)

Step 3 $B^G \rightarrow B$ finite $\Rightarrow B = B_1 \times \dots \times B_r$ product of strictly henselian local rings.

$\Rightarrow$ another reduction where B is s.h. local, G fixes x, y while acting trivially on residue fields $\pi(x) = \pi(y)$.

$$\begin{array}{ccccc} \text{Spec } A & \xrightarrow[\text{scale on } x]{f} & \text{Spec } B & & \\ \downarrow x & \xleftarrow{s} & \downarrow y & \xrightarrow{\text{open and closed immersion}} & \\ \text{Im}(s) \subseteq [\text{Spec } A/G] & \rightarrow & [\text{Spec } B/G] & & \\ \downarrow \pi_A & \xleftarrow{s} & \downarrow \pi_B & & \\ \text{Spec } B^G & \rightarrow & \text{Spec } A^G & \rightarrow & \text{Spec } B^G \\ & \searrow \pi & & & \end{array}$$

Theorem. (Keel - Mori Theorem)

$\mathcal{X}$ = DM stack separated and finite type / Noetherian alg. space S .

Then $\exists$ c.m.s. $\pi: \mathcal{X} \rightarrow X$ with $\mathcal{O}_{\mathcal{X}} = \pi_* \mathcal{O}_{\mathcal{X}}$ s.t.

(1) X separated & f.t. / S

(2) π proper universal homeo

(3) $\forall$ flat morphism $X' \rightarrow X$ of alg. spaces, $\mathcal{X} \times_X X' \rightarrow X'$ c.m.s.

Pf. I first show (Existence of c.m.s.)

- can assume $S = \text{Spec}(R)$ affine.
- question Zariski-local on $\mathcal{X}$. $\Rightarrow$ suffices to show $\forall$ cl point $\overset{x}{V}$ has open nbhd that has c.m.s.

$\exists f: (\mathcal{W} = [\text{Spec } A / G_{\mathcal{W}}], w) \rightarrow (\mathcal{X}, \pi)$.

(iso on geo. stabilizer group on w)

Claim. locus $\overset{w}{V}$ in $|\mathcal{W}|$ where map of stabilizer groups is bty. is open.

Pf. Fiber of $I_{\mathcal{W}} \rightarrow I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W}$ over $z \in \mathcal{W}(k)$ is $G_z \rightarrow G_{\mathcal{X}(z)}$.

$$\begin{array}{ccc} I_{\mathcal{W}} & \xrightarrow[\text{cl}]{\text{open}} & I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W} \\ \downarrow \square & & \downarrow \\ \mathcal{W} & \xrightarrow[\text{cl}]{\text{open}} & \mathcal{W} \times_{\mathcal{X}} \mathcal{W} \end{array} \Rightarrow$$

$$A_2(|I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W}| \setminus |I_{\mathcal{W}}|) \overset{\text{cl}}{\subseteq} \mathcal{W} \\ \overset{w}{=} \mathcal{W}^c$$

$\mathcal{W} \rightarrow \mathcal{X}$
 $\square$
 $\mathcal{W}$ is affine

$\square$

$$\pi_w : \mathcal{W} \rightarrow W = \text{Spec } A^{G_n} \text{ c.m.s.}$$

$$\text{Take } \pi_w(w) \in X_1 = D(h) \subseteq \pi_w(W).$$

$$\Rightarrow \pi_1 = \pi_w^{-1}(X_1) \rightarrow X_1 \text{ identified as } [\text{Spec } A_1 / G_n] \rightarrow \text{Spec } A_1^{G_n}$$

This gives an affine étale morphism $g: (\pi_1 = [\text{Spec } A_1 / G_n], w)$

in which all maps of stabilizer groups are bij. $\rightarrow (\pi_1, \pi)$

Now we show $\pi_0 = \text{Im}(g)$ admits a c.m.s.

$$\begin{array}{ccccc} \text{open substack} & & & & \\ \downarrow & & & & \\ \text{Spec } A_2 / G & \xrightarrow{\pi_1 \times \pi_0} & \text{Spec } A_1 / G & \xrightarrow{g} & \pi_0 \\ \downarrow \text{c.m.s.} & \searrow & \downarrow \text{c.m.s.} & & \downarrow \exists \\ X_2 & \xRightarrow{\cong} & X_1 & \longrightarrow & X_0 := X_1 / X_2 \\ \text{Spec } A_2^{G_n} & & \text{Spec } A_1^{G_n} & & \end{array}$$

g induces bij. of st groups

$\Rightarrow$ same holds for $\pi_2 \Rightarrow \pi_1$

$\Rightarrow$ by Prop., $X_2 \Rightarrow X_1$ étale. (étale groupoid structure)

$\Rightarrow |\pi_2| \rightarrow |\pi_1| \times |\pi_1|$ inj of cl points,

so $X_2 \Rightarrow X_1$ étale equivalence relation.

$$\begin{array}{ccccc} & \xrightarrow{\text{surj. étale}} & & & \\ \pi_2 & & \pi_1 & & \\ \downarrow & \square & \downarrow & \text{c.m.s.} & \\ \pi_1 & \xrightarrow{\quad} & \pi_0 & \xrightarrow{\quad} & \\ \downarrow & \square & \downarrow & & \\ X_2 & \xrightarrow{\quad} & X_1 & \longrightarrow & \\ \downarrow & \square & \downarrow & & \\ X_1 & \longrightarrow & X_0 & & \end{array}$$

$$\pi_0 \rightarrow X_0 \text{ c.m.s.}$$

Conditions:

$$\begin{array}{ccc} \pi & \xrightarrow{\text{proper}} & X \\ \searrow & & \downarrow \\ & S & \Rightarrow \text{sep} \end{array}$$



Cor. (Local structure of c.m.s.)

$\mathcal{X}$ = DM stack f.t. sep / Noetherian alg. space S ,
 $\pi: \mathcal{X} \rightarrow X$ c.m.s.

$$\begin{array}{ccc} \forall \text{ c.p. } x \in |\mathcal{X}|: & \exists [\operatorname{Spec} A/G_n] & \longrightarrow \mathcal{X} \\ & \downarrow \quad \square \quad \downarrow \pi & \\ & \operatorname{Spec} A^{G_n} & \longrightarrow X \end{array}$$

s.t. $\operatorname{Spec} A^{G_n} \rightarrow X$ is étale nbhd of $\pi(x)$.

Prop. $\mathcal{X}$ DM stack f.t. sep / Noetherian alg. space S ,
 $\pi: \mathcal{X} \rightarrow X$ c.m.s.

If $\mathcal{I}$ LB on $\mathcal{X}$, then for suff. divisible N
 $\mathcal{I}^{\otimes N}$ descends to X .